Journal of Global Optimization, **29:** 439–453, 2004.

Nonsmooth Variational Problems in the Limit Case and Duality

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(Received 1 August 2003; accepted 13 August 2003)

Abstract. The paper contains a duality result and two existence theorems for nonsmooth boundary value problems, with unbounded constraints, in the limit case. Examples illustrate the abstract results.

Mathematics Subject Classifications. 49J40, 35J85, 35A15.

Key words. duality, minimax principle, nonsmooth analysis, variational–hemivariational inequalities.

1. Introduction

In this paper we discuss the so-called limit case of the minimax principle in the nonsmooth critical point theory from the point of view of effective applications to nonsmooth boundary value problems.

Let $f\colon X \to [-\infty, +\infty]$ be a function (the values $\pm\infty$ are admitted) on a real reflexive Banach space X. Consider a compact topological submanifold Q of X with nonempty boundary ∂Q (in the sense of manifolds) and a nonempty closed subset S of X. Corresponding to the sets Q and S we introduce the numbers

$$a := \inf_S f, \tag{1}$$

$$b := \sup_{\gamma^* \in \Gamma^*} \inf_{x \in S} f(\gamma^*(x)), \tag{2}$$

$$c := \inf_{\gamma \in \Gamma} \sup_{x \in Q} f(\gamma(x)), \tag{3}$$

where

$$\Gamma^* = \{\gamma^* \in C(X, X)\colon\ \gamma^* \text{ homeomorphism},\ \gamma^*|_{\partial Q} = id_{\partial Q}\} \tag{4}$$

and

$$\Gamma = \{\gamma \in C(Q, X)\colon\ \gamma|_{\partial Q} = id_{\partial Q}\}. \tag{5}$$

We note from (1), (2) and (4) that $a \leqslant b$. In order to compare b and c we assume the following linking condition for Q and S:

$$\partial Q \cap S = \emptyset \quad \text{and} \quad \gamma(Q) \cap S \neq \emptyset, \quad \forall \gamma \in \Gamma. \tag{6}$$

Then from (2)–(6) we see that $b \leqslant c$. Indeed, for arbitrary elements $\gamma \in \Gamma$ and $\gamma^* \in \Gamma^*$ we have that $(\gamma^*)^{-1} \circ \gamma \in \Gamma$ and there is some $z \in Q$ with $(\gamma^*)^{-1}(\gamma(z)) \in S$. It follows that

$$\inf_{x \in S} f(\gamma^*(x)) \leqslant f(\gamma^*((\gamma^*)^{-1}(\gamma(z)))) = f(\gamma(z)) \leqslant \sup_{x \in Q} f(\gamma(x)),$$

which yields $b \leqslant c$. Therefore one has

$$a \leqslant b \leqslant c. \tag{7}$$

An important feature of relation (7) is that the number b can be viewed as a dual expression of number c (see (2), (3)). This duality will be exploited in Section 2. The situation of equality $a = c$ in (7) (a fortiori, $a = b = c$) is called the limit case. The nonsmooth boundary value problems studied in Section 3 address this case.

Our main results deal with a function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ satisfying the structure hypothesis

(H_f) $f = \Phi + \alpha$, where $\Phi\colon X \to \mathbb{R}$ is locally Lipschitz and $\alpha\colon X \to \mathbb{R} \cup \{+\infty\}$ is convex, proper (i.e., $\not\equiv +\infty$) and lower semicontinuous.

To develop our duality approach, we are concerned in Section 2 also with functionals $g\colon X \to \mathbb{R} \cup \{-\infty\}$ satisfying

($\widetilde{\mathrm{H}}_g$) $g = \Psi + \beta$, where $\Psi\colon X \to \mathbb{R}$ is locally Lipschitz and $\beta\colon X \to \mathbb{R} \cup \{-\infty\}$ is concave, proper (i.e., $\not\equiv -\infty$) and upper semicontinuous.

For the class of nonsmooth functionals (H_f) we give the basic notions of critical point and Palais-Smale condition.

DEFINITION 1 (Motreanu and Panagiotopoulos [8], p. 64). An element $u \in X$ is called a critical point of $f = \Phi + \alpha\colon X \to \mathbb{R} \cup \{+\infty\}$ satisfying (H_f) if

$$\Phi^0(u; v - u) + \alpha(v) - \alpha(u) \geqslant 0, \quad \forall v \in X.$$

The notation Φ^0 stands for the generalized directional derivative of Φ in the sense of Clarke [5], p. 25, that is

$$\Phi^0(u; v) = \limsup_{w \to u,\, t \to 0^+} \frac{1}{t}(\Phi(w + tv) - \Phi(w)), \quad \forall u, v \in X.$$

DEFINITION 2 (Marano and Motreanu [7]). The functional $f = \Phi + \alpha$: $X \to \mathbb{R} \cup \{+\infty\}$ satisfying (H_f) verifies the Palais-Smale condition around the set $S \subset X$ at level $r \in \mathbb{R}$ if

$(\mathrm{PS})_{f,S,r}$ Every sequence $\{u_n\}$ in X such that $d(u_n, S) \to 0$, $f(u_n) \to r$ and

$$\Phi^0(u_n; v - u_n) + \alpha(v) - \alpha(u_n) \geqslant -\varepsilon_n \|v - u_n\|, \quad \forall n \geqslant 1, \ v \in X,$$

for some $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0^+$, contains a (strongly) convergent subsequence.

If $\alpha = 0$, Definitions 1 and 2 reduce to the corresponding notions in the critical point theory for locally Lipschitz functions as introduced by Chang [4]. If $\Phi \in C^1(X)$ and α is as in (H_f), Definitions 1 and 2 become the ones in the nonsmooth critical point theory of Szulkin [12].

Some further notations are needed. For any $r \in \mathbb{R}$, we denote $f_r = \{x \in X: f(x) \leqslant r\}$ and $f^r = \{x \in X: f(x) \geqslant r\}$. For a function f: $X \to \mathbb{R} \cup \{+\infty\}$ satisfying (H_f) the set of critical points (in the sense of Definition 1) at level $r \in \mathbb{R}$ is denoted by $K_r(f)$, that is

$$K_r(f) = \{u \in X: f(u) = r \text{ and } u \text{ is a critical point of } f \text{ in the sense of Definition 1}\}.$$

For any $\delta > 0$, the closed δ-neighborhood of the set S in X is denoted by $N_\delta(S)$, i.e. $N_\delta(S) = \{x \in X: d(x, S) \leqslant \delta\}$. The domain of the convex function α: $X \to \mathbb{R} \cup \{+\infty\}$ in (H_f) is denoted D_α, i.e., $D_\alpha = \{x \in X: \alpha(x) < +\infty\}$.

The minimax principle in the limit case (i.e., $c = a$ in (7)) for the functionals satisfying (H_f) is the following.

THEOREM 1 (Marano and Motreanu [7]). *Suppose that the conditions* (H_f) *and* (6) *hold. If, in addition,*

(f_1) $\sup_Q f < +\infty$ *and* $\partial Q \subset f_a$;
(f_2) $c = a$;
(f_3) $(\mathrm{PS})_{f,S,a}$;
(f_4) $N_{\varepsilon_0}(S) \subset D_\alpha$ *and the set* $N_\delta(S) \cap f^{a-\delta} \cap f_{a+\delta}$ *is closed*, $\forall \delta \in]0, \varepsilon_0[$, *for some* $\varepsilon_0 > 0$,

then one has $K_a(f) \cap S \neq \emptyset$.

Notice that under the assumptions of Theorem 1 relation (7) becomes the limit case

$$a = b = c \in \mathbb{R} \tag{8}$$

and the common value in (8) is a critical value of f, i.e. there exists a critical point $u \in X$ of f satisfying $f(u) = a$. Moreover, Theorem 1 provides the important information that the critical point $u \in X$ is located on S.

The rest of the paper is organized as follows. In Section 2, by weakening the assumption (f_2) to have $b = a$, we present a minimax principle ensuring that b (the "dual" value to c) is a critical value of f. This can be viewed as a dual result with respect to Theorem 1. Section 3 is devoted to effective applications of the minimax principle in the limit case $c = a$ to boundary value problems with discontinuous nonlinearities and unbounded constraints both in non-resonant and resonant cases.

2. A Dual Minimax Principle

To establish a minimax result, dual to Theorem 1, in the case $a = b$, we need the deformation lemma in [7] for functions $g: X \to \mathbb{R} \cup \{-\infty\}$ belonging to the class ($\widetilde{\mathrm{H}}_g$) (see Section 1). Some preliminaries are necessary. Given $d \in \mathbb{R}$ and the function $g = \Psi + \beta: X \to \mathbb{R} \cup \{-\infty\}$ satisfying ($\widetilde{\mathrm{H}}_g$) we denote

$$\widetilde{K}_d(g) := \{u \in X: g(u) = d \text{ and } \Psi^0(u; u - v) + \beta(u) - \beta(v) \geqslant 0, \ \forall v \in X\}$$

and $D_\beta := \{x \in X: \beta(x) > -\infty\}$.

We say that a function $g: X \to \mathbb{R} \cup \{-\infty\}$ satisfying ($\widetilde{H}_g$) verifies the condition $(\widetilde{\mathrm{PS}})_{g,B,d}$ for a subset $B \subset X$ and a number $d \in \mathbb{R}$ if

$(\widetilde{\mathrm{PS}})_{g,B,d}$ Each sequence $\{x_n\} \subset X$ such that $d(x_n, B) \to 0$, $g(x_n) \to d$ and

$$\Psi^0(x_n; x_n - x) + \beta(x_n) - \beta(x) \geqslant -\varepsilon_n \|x_n - x\|, \quad \forall n \geqslant 1, \ x \in X,$$

where $\varepsilon_n \to 0^+$, possesses a (strongly) convergent subsequence.

In the sequel we need the following deformation result.

LEMMA 1 (Marano and Motreanu [7]). *Let a function* $g = \Psi + \beta: X \to \mathbb{R} \cup \{-\infty\}$, *two nonempty closed subsets A, B of X and a number* $d \in \mathbb{R}$ *satisfy* ($\widetilde{H}_g$), $(\widetilde{\mathrm{PS}})_{g,B,d}$,

(g_1) $A \cap B = \emptyset$, $A \subset g^d$, $B \subset g_d$, $\widetilde{K}_d(g) \cap B = \emptyset$,
(g_2) *there exists* $\varepsilon_0 > 0$ *such that* $N_{\varepsilon_0}(B) \subset D_\beta$ *and the set* $N_\delta(B) \cap g^{d-\delta} \cap g_{d+\delta}$ *is closed,* $\forall \delta \in]0, \varepsilon_0[$.

Then there exist $\varepsilon > 0$ *and a homeomorphism* $\eta\colon X \to X$ *with the properties*:

(i) $\eta(x) = x, \forall x \in A$;
(ii) $\eta(B) \subset g_{d-\varepsilon}$.

We state now our minimax principle in the case $a = b$ (see (1), (2)).

THEOREM 2. *Assume that the function* $f\colon X \to \mathbb{R} \cup \{+\infty\}$, *the compact topological submanifold* Q *of* X *with nonempty boundary* ∂Q (*in the sense of manifolds*) *and the nonempty closed subset* S *of* X *satisfy* (H_f), (6), (f_1), (f_3), (f_4) *and*

(f_2') $a = b$.

Then one has $K_a(f) \cap S \neq \emptyset$.

Proof. First we note that thanks to (f_1) and (f_2') we have that $a = b \in \mathbb{R}$. Arguing by contradiction, suppose that $K_a(f) \cap S = \emptyset$. Consider the function $g = -f\colon X \to \mathbb{R} \cup \{-\infty\}$. Since f verifies (H_f), then g satisfies $(\widetilde{H}_g)$, with $\Psi := -\Phi$ and $\beta := -\alpha$.

Let $A = \partial Q$, $B = S$ and $d = -a$. To check $(\widetilde{\mathrm{PS}})_{g,B,d}$, let $\{x_n\} \subset X$ be a sequence such that $d(x_n, B) \to 0$, $g(x_n) \to d$ and

$$\Psi^0(x_n; x_n - x) + \beta(x_n) - \beta(x) \geqslant -\varepsilon_n \|x_n - x\|, \quad \forall n \geqslant 1,\ x \in X,$$

with $\varepsilon_n \to 0^+$. These read as $d(x_n, S) \to 0$, $f(x_n) \to a$ and

$$\Phi^0(x_n; x - x_n) + \alpha(x) - \alpha(x_n) \geqslant -\varepsilon_n \|x - x_n\|, \quad \forall n \geqslant 1,\ x \in X.$$

By (f_3), we infer that the sequence $\{x_n\}$ has a strongly convergent subsequence, so property $(\widetilde{\mathrm{PS}})_{g,B,d}$ holds.

By (f_1), we have that $\partial Q \subset g^d$. Since $S \subseteq f^a$ it follows that $S \subset g_d$. Moreover, $\widetilde{K}_d(g) \cap B = \emptyset$ because $K_a(f) \cap S = \emptyset$ and $\widetilde{K}_d(g) = K_a(f)$. Thus condition (g_1) is verified. Since the set

$$N_\delta(B) \cap g^{d-\delta} \cap g_{d+\delta} = N_\delta(B) \cap f^{a-\delta} \cap f_{a+\delta}, \quad \forall \delta \in]0, \varepsilon_0[,$$

is closed in view of assumption (f_4), condition (g_2) is fulfilled.

Consequently, we can apply Lemma 1. We find a number $\varepsilon > 0$ and a homeomorphism $\eta\colon X \to X$ such that

(i) $\eta(x) = x, \forall x \in \partial Q$;
(ii) $\eta(S) \subset g_{d-\varepsilon}$.

Assertion (i) implies that $\eta \in \Gamma^*$. Property (ii) expresses that

$$f(\eta(x)) \geqslant a + \varepsilon, \quad \forall x \in S.$$

Since $\eta \in \Gamma^*$, by (f_2') we obtain

$$a = \sup_{\gamma^* \in \Gamma^*} \inf_{x \in S} f(\gamma^*(x)) \geqslant \inf_{x \in S} f(\eta(x)) \geqslant a + \varepsilon.$$

This contradiction completes the proof. □

Remark 1. Taking into account the definitions of b and c in (2) and (3), respectively, Theorem 2 can be regarded as a result dual to Theorem 1. Theorem 2 extends from the locally Lipschitz case to the class (H_f) the part in Theorem 3.1 of Barletta and Marano [2] addressing the situation $a = b$ and with the linking property considered here. Theorem 2 extends Theorem 1 because assumption (f_2') is more general than condition (f_2) (see (7)).

3. Applications to Boundary Value Problems

We turn now to the application of Theorem 1 to boundary value problems. These will be formulated in terms of variational-hemivariational inequalities. For the nonsmooth variational theory of variational-hemivariational inequalities we refer to Motreanu and Panagiotopoulos [8]. Different other results and applications of hemivariational inequalities can be found in Gao [6], Naniewicz and Panagiotopoulos [9], Panagiotopoulos [10].

Let Ω be a nonempty, bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, with a C^1 boundary $\partial\Omega$. The Hilbert space $H_0^1(\Omega)$ is endowed with the scalar product

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H_0^1(\Omega),$$

and the induced norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2 dx \right)^{\frac{1}{2}}, \quad \forall u \in H_0^1(\Omega).$$

Due to the continuity of embedding $H_0^1(\Omega) \subset L^p(\Omega)$ for $1 \leqslant p \leqslant 2^* = \frac{2N}{N-2}$, there is a constant $c_p > 0$ such that

$$\|u\|_{L^p(\Omega)} \leqslant c_p \|u\|, \quad \forall u \in H_0^1(\Omega). \tag{9}$$

The embedding is compact for $1 \leqslant p < 2^*$.

Consider the sequence of eigenvalues of $-\Delta$ on $H_0^1(\Omega)$

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$$

and a corresponding sequence $\{\varphi_j\}$ of eigenfunctions

$$\begin{cases} -\Delta\varphi_j = \lambda_j \varphi_j & \text{in } \Omega \\ \varphi_j = 0 & \text{on } \partial\Omega \end{cases}$$

normalized as follows $\|\varphi_j\|^2 = 1 = \lambda_j \|\varphi_j\|^2_{L^2(\Omega)}$, $\forall j \geqslant 1$ (see, e.g., Brézis [3]).

Let a positive integer k be fixed such that $\lambda_k < \lambda_{k+1}$. We denote

$$V = \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}, \quad V^\perp = \{w \in H_0^1(\Omega)\colon (w, v) = 0, \ \forall v \in V\}.$$

Let α: $H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous, proper functional, let h: $H_0^1(\Omega) \to \mathbb{R}$ be a locally Lipschitz function and let $\lambda \in]\lambda_k, \lambda_{k+1}[$ be a fixed number. Consider the following (non-resonant) variational-hemivariational inequality problem:

(P_1) Find $u \in D_\alpha \subset H_0^1(\Omega)$ such that

$$-\int_\Omega \nabla u(x) \cdot \nabla(v-u)(x) \mathrm{d}x + \lambda \int_\Omega u(x)(v(x) - u(x)) \mathrm{d}x$$
$$\leqslant h^0(u; v-u) + \alpha(v) - \alpha(u), \quad \forall v \in D_\alpha.$$

We assume that α and h satisfy:

(j_1) D_α is closed and there exist $r > 0$ and $0 < \varepsilon < r$ such that

$$\{u \in H_0^1(\Omega)\colon r - \varepsilon < \|u\| < r + \varepsilon\} \subset D_\alpha;$$

(j_2) $h(u) + \alpha(u) \geqslant -\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) r^2$, $\forall u \in V^\perp$, $\|u\| = r$, with $r > 0$ prescribed in (j_1);

(j_3) there exists $\rho > r$, for $r > 0$ in (j_1), such that for all $u = u_1 + t\varphi_{k+1}$, $u_1 \in V$, $\|u_1\| \leqslant \rho$, $t \in [0, \rho]$ one has

$$h(u) + \alpha(u) \leqslant \frac{1}{2}\left(\frac{\lambda}{\lambda_k} - 1\right)\|u_1\|^2 - \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) t^2;$$

(j_4) $\limsup_{n \to \infty} h^0(u_n; u - u_n) \leqslant 0$ whenever $u_n \rightharpoonup u$ in $H_0^1(\Omega)$.

Our result in studying problem (P_1) is the following.

THEOREM 3. *Assume* (j_1)–(j_4). *Then problem* (P_1) *has at least a solution* $u \in H_0^1(\Omega)$ *satisfying* $u \in V^\perp$ *and* $\|u\| = r$. *In addition, we have*

$$(\lambda/2)\|u\|^2_{L^2(\Omega)} - h(u) - \alpha(u) = r^2/2.$$

Proof. Consider the functional $f = \Phi + \alpha$: $H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$, with Φ: $H_0^1(\Omega) \to \mathbb{R}$ given by

$$\Phi(v) = \frac{1}{2}\left(\|v\|^2 - \lambda\|v\|^2_{L^2(\Omega)}\right) + h(v), \quad \forall v \in H_0^1(\Omega). \tag{10}$$

Since Φ is locally Lipschitz, the structure of $f=\Phi+\alpha$ complies with hypothesis (H_f).

With ρ and r fixed by hypotheses (j_1)–(j_3), we define

$$Q=\left(V\cap\overline{B}_\rho\right)\oplus[0,\rho\varphi_{k+1}] \quad\text{and}\quad S=\partial B_r\cap V^\perp, \tag{11}$$

where $B_r=\{v\in H_0^1(\Omega)\colon \|v\|<r\}$ and $\partial B_r=\{v\in H_0^1(\Omega)\colon \|v\|=r\}$.

Since $r<\rho$, the compact topological manifold Q and the closed set S satisfy (6) (see Ambrosetti [1, Lemma 4.1] or Rabinowitz [11, Proposition 5.9]). Every $u\in Q$ can be expressed as $u=u_1+u_2$, with $u_1=\sum_{i=1}^k t_i\varphi_i\in V$ and $u_2=t\varphi_{k+1}$, where $t_1,\dots,t_k\in\mathbb{R}$, $\|u_1\|\leqslant\rho$, $t\in[0,\rho]$. Then using (10) and (j_3) we have

$$\begin{aligned} f(u)&=\frac{1}{2}\sum_{i=1}^{k}\left(1-\frac{\lambda}{\lambda_i}\right)t_i^2+\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)t^2+h(u)+\alpha(u)\\ &\leqslant\frac{1}{2}\left(1-\frac{\lambda}{\lambda_k}\right)\|u_1\|^2+\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)t^2+h(u)+\alpha(u)\leqslant 0.\end{aligned}$$

Thus we have shown that $Q\subset f_0$, hence $\partial Q\subset f_0$, which ensures (f_1) with $a=0$.

Taking into account (11), if $u\in S$ we have that $\|u\|=r$ and $u=\sum_{i=k+1}^{+\infty}t_i\varphi_i$, with $t_i\in\mathbb{R}$, $\forall i\geqslant k+1$. Then using (10) and (j_2), it results

$$f(u)=\frac{1}{2}\sum_{i=k+1}^{+\infty}\left(1-\frac{\lambda}{\lambda_i}\right)t_i^2+h(u)+\alpha(u)\geqslant\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)r^2+h(u)+\alpha(u)\geqslant 0.$$

By (1), this means that $a=\inf_S f\geqslant 0$. In view of (3) and (7), we find that

$$0\leqslant a\leqslant c=\inf_{\gamma\in\Gamma}\sup_{z\in Q}f(\gamma(z))\leqslant\sup_{z\in Q}f(z)\leqslant 0,$$

so (f_2) is satisfied with $a=c=0$.

To show (f_3), i.e. $(\mathrm{PS})_{f,S,a}$ with $a=0$, let the sequence $\{u_n\}\subset H_0^1(\Omega)$ satisfy $d(u_n,S)\to 0$, $f(u_n)\to 0$ and

$$\Phi^0(u_n;v-u_n)+\alpha(v)-\alpha(u_n)\geqslant-\varepsilon_n\|v-u_n\|,\quad\forall n\geqslant 1,\ v\in D_\alpha, \tag{12}$$

where $\varepsilon_n\to 0^+$. Since $d(u_n,S)\to 0$ and S is a bounded set, the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Then, along a relabelled subsequence, we may assume that $u_n\rightharpoonup u$ in $H_0^1(\Omega)$ and $u_n\to u$ in $L^2(\Omega)$, with $u\in D_\alpha$ (since $u_n\in D_\alpha$ and, by (j_1), D_α is a closed convex set). Setting $v=u$ in (12) we derive that

$$\begin{aligned}\|u_n\|^2\leqslant&\int_\Omega\nabla u_n(x)\cdot\nabla u(x)\mathrm{d}x-\lambda\int_\Omega u_n(x)(u(x)-u_n(x))\mathrm{d}x+\\ &+h^0(u_n;u-u_n)+\alpha(u)-\alpha(u_n)+\varepsilon_n\|u_n-u\|,\quad\forall n\geqslant 1.\end{aligned}$$

Using (j_4) and the lower semicontinuity of α we can pass to the limit for obtaining

$$\limsup_{n\to+\infty}\|u_n\|^2 \leqslant \|u\|^2+\limsup_{n\to+\infty} h^0(u_n;u-u_n)+\alpha(u)-\liminf_{n\to+\infty}\alpha(u_n)\leqslant \|u\|^2.$$

This ensures that $u_n \to u$ in $H_0^1(\Omega)$, thus (f_3) is verified (with $a=0$).

Taking $0<\varepsilon_0<\varepsilon$ (with ε in (j_1)), we obtain from (j_1) that $N_{\varepsilon_0}(S)\subset \operatorname{int}D_\alpha$. Moreover, for any $\delta\in]0,\varepsilon_0[$ we have that $N_\delta(S)\cap f^{-\delta}\cap f_\delta$ is closed in $H_0^1(\Omega)$ since $\alpha|_{\operatorname{int}D_\alpha}$ is continuous. Thus (f_4) holds true.

We may apply Theorem 1. The proof is complete by pointing out that every critical point of the functional $f=\Phi+\alpha$, with Φ given in (10), is a solution to problem (P_1) satisfying $f(u)=0$ and the location property $u\in S=\partial B_r\cap V^\perp$. □

Remark 2. The above proof ensures that for every $s\in]0,r[$ (with r in (j_1)) there exists a solution u_s of (P_1) lying in $\partial B_s\cap V^\perp$. Therefore, actually this problem possesses infinitely (even uncountably) many nontrivial solutions inside $B_r\cap V^\perp$.

Remark 3. Theorem 3 remains valid if we assume $\lambda\in[\lambda_k,\lambda_{k+1}]$. The proof is the same.

We provide an example of applying Theorem 3. We use the notation

$$W=\operatorname{span}\{\varphi_1,\ldots,\varphi_k,\varphi_{k+1}\}.$$

EXAMPLE 1. Let $J_1, J_2\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be functions such that $J_1(\cdot,t), J_2(\cdot,t)\colon \Omega\to\mathbb{R}$ are measurable on Ω for each $t\in\mathbb{R}$, $J_1(x,\cdot), J_2(x,\cdot):\mathbb{R}\to\mathbb{R}$ are locally Lipschitz for a.e. $x\in\Omega$, $J_1(\cdot,0), J_2(\cdot,0)\in L^1(\Omega)$. Assume that

$$\int_\Omega J_1(x,0)\,\mathrm{d}x=-\int_\Omega J_2(x,0)\,\mathrm{d}x\geqslant 0, \tag{13}$$

$$|z|\leqslant C(1+|t|^{p-1}),\quad \forall z\in\partial J_1(x,t)\cup\partial J_2(x,t)\quad \text{a.e. } x\in\Omega,\quad \forall t\in\mathbb{R}, \tag{14}$$

for some constants $C\geqslant 0$ and $2<p<2^*$,

$$J_1(x,t)\leqslant \frac{1}{2}\left(\frac{\lambda}{\lambda_k}-1\right)\lambda_1 t^2\quad \text{a.e. } x\in\Omega,\quad \forall t\in\mathbb{R}, \tag{15}$$

$$J_2(x,t)\geqslant -\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\lambda_{k+2} t^2\quad \text{a.e. } x\in\Omega,\quad \forall t\in\mathbb{R}, \tag{16}$$

with $\lambda\in]\lambda_k,\lambda_{k+1}[$.

Define the function $h\colon H_0^1(\Omega)\to\mathbb{R}$ by

$$h(u)=\int_\Omega J_1(x,u_1(x))\,\mathrm{d}x-\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\|u_2\|^2+\int_\Omega J_2(x,u_3(x))\,\mathrm{d}x,$$

for all $u=u_1+u_2+u_3\in H_0^1(\Omega)$ with $u_1\in V, u_2\in\mathbb{R}\varphi_{k+1}$ and $u_3\in W^\perp$. Taking into account (14), the function h: $vH_0^1(\Omega)\to\mathbb{R}$ is locally Lipschitz.

Let K be a closed, convex subset of $H_0^1(\Omega)$ such that

$$W\oplus\{u\in W^\perp\colon \|u\|\leqslant r_0\}\subset K,$$

for some $r_0>0$, and let $\alpha=I_K$: $H_0^1(\Omega)\to\mathbb{R}\cup\{+\infty\}$ denote the indicator function of K, i.e.

$$I_K(u)=\begin{cases}0 & \text{if } \ u\in K\\ +\infty & \text{otherwise.}\end{cases}$$

We claim that conditions $(\mathrm{j}_1)-(\mathrm{j}_4)$ in Theorem 3 are verified.

Fix an arbitrary number $r\in]0,r_0[$ and any $0<\varepsilon<\min\{r_0-r,r\}$. Condition (j_1) is satisfied since $\overline{B}_{r+\varepsilon}\subset B_{r_0}\subset K=D_\alpha$ and D_α is closed.

By (13), (16) and the variational characterization of λ_{k+2}, it follows that

$$\begin{aligned}h(u)+\alpha(u)&\geqslant-\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\|u_2\|^2-\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\lambda_{k+2}\|u_3\|^2_{L^2(\Omega)}\\&\geqslant-\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)(\|u_2\|^2+\|u_3\|^2)=-\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)r^2,\end{aligned}$$

for every $u=u_2+u_3\in V^\perp$ with $u_2\in\mathbb{R}\varphi_{k+1}, u_3\in W^\perp$ and $\|u\|=r$. This shows that (j_2) is true.

Relations (13) and (9) with the constant $c_2=\frac{1}{\sqrt{\lambda_1}}$ imply that for every $u=u_1+u_2\in W$ with $u_1\in V, u_2\in\mathbb{R}\varphi_{k+1}$, we have

$$\begin{aligned}h(u)+\alpha(u)&\leqslant\frac{1}{2}\left(\frac{\lambda}{\lambda_k}-1\right)\lambda_1\|u_1\|^2_{L^2(\Omega)}-\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\|u_2\|^2\\&\leqslant\frac{1}{2}\left(\frac{\lambda}{\lambda_k}-1\right)\|u_1\|^2-\frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\|u_2\|^2.\end{aligned}$$

Condition (j_3) is verified with an arbitrary $\rho>r$.

It remains to check (j_4). Let $\{u_n\}\subset H_0^1(\Omega)$ be a sequence such that $u_n\rightharpoonup u$ in $H_0^1(\Omega)$, for some $u\in H_0^1(\Omega)$. Writing $u=u^1+u^2+u^3, u_n=u_n^1+u_n^2+u_n^3$, with $u^1,u_n^1\in V, u^2,u_n^2\in\mathbb{R}\varphi_{k+1}, u^3,u_n^3\in W^\perp$, we see that $u_n^1\rightharpoonup u^1, u_n^2\to u^2, u_n^3\rightharpoonup u^3$ in $H_0^1(\Omega)$. Due to the growth condition in (14), we may apply Aubin-Clarke theorem (see Clarke [5], pp. 83–85). We obtain that

$$\begin{aligned}h^0(u_n;u-u_n)\leqslant&\int_\Omega J_1^0(x,u_n^1(x);u^1(x)-u_n^1(x))\,\mathrm{d}x-\\&-\left(1-\frac{\lambda}{\lambda_{k+1}}\right)(u_n^2,u^2-u_n^2)+\int_\Omega J_2^0(x,u_n^3(x);u^3(x)-u_n^3(x))\mathrm{d}x.\end{aligned}$$

Passing to lim sup as $n\to+\infty$ we have that

$$\limsup_{n\to+\infty} h^0(u_n;u-u_n)\leqslant\limsup_{n\to+\infty}\int_\Omega J_1^0(x,u_n^1(x);u^1(x)-u_n^1(x))\,\mathrm{d}x+ \\ +\limsup_{n\to+\infty}\int_\Omega J_2^0(x,u_n^3(x);u^3(x)-u_n^3(x))\,\mathrm{d}x. \qquad (17)$$

By the compactness of the embedding $H_0^1(\Omega)\subset L^p(\Omega)$, along a relabelled subsequence we may suppose that $u_n^1\to u^1, u_n^3\to u^3$ in $L^p(\Omega), u_n^1(x)\to u^1(x)$, $u_n^3(x)\to u^3(x)$ a.e. $x\in\Omega$ and we can find a function $g\in L^p(\Omega)$ such that $|u_n^1(x)|\leqslant g(x), |u_n^3(x)|\leqslant g(x)$ a.e. $x\in\Omega$. Then, using (14) we have the estimate

$$\begin{aligned}|J_1^0(x,u_n^1(x);u^1(x)-u_n^1(x))| &\leqslant \max_{\zeta\in\partial J_1(x,u_n^1(x))}|\zeta||u^1(x)-u_n^1(x)| \\ &\leqslant C(1+|u_n^1(x)|^{p-1})|u^1(x)-u_n^1(x)| \\ &\leqslant C(1+g(x)^{p-1})(|u^1(x)|+g(x)) \\ &\qquad \text{a.e. } x\in\Omega, \forall\ n\geqslant 1.\end{aligned}$$

Similarly, we get

$$\begin{aligned}&|J_2^0(x,u_n^3(x);u^3(x)-u_n^3(x))| \\ &\quad\leqslant C(1+g(x)^{p-1})(|u^3(x)|+g(x)) \text{ a.e. } x\in\Omega,\ \forall\ n\geqslant 1.\end{aligned}$$

The estimates above allow to make use of Fatou's lemma in (17). This leads to

$$\limsup_{n\to+\infty} h^0(u_n;u-u_n)\leqslant\int_\Omega \limsup_{n\to+\infty} J_1^0(x,u_n^1(x);u^1(x)-u_n^1(x))\mathrm{d}x+ \\ +\int_\Omega \limsup_{n\to+\infty} J_2^0(x,u_n^3(x);u^3(x)-u_n^3(x))\,\mathrm{d}x.$$

The upper semicontinuity of $J_1^0(x,\cdot;\cdot)$ and $J_2^0(x,\cdot;\cdot)$ ensure that assertion (j_4) is verified. Thus Theorem 3 can be applied.

The rest of the Section is devoted to a resonant problem. Let J: $\Omega\times\mathbb{R}\to\mathbb{R}$ be a function such that $J(\cdot,t)$: $\Omega\to\mathbb{R}$ is measurable for each $t\in\mathbb{R}$, $J(x,\cdot)$: $\mathbb{R}\to\mathbb{R}$ is locally Lipschitz for a.e. $x\in\Omega$ whose generalized gradient $\partial J(x,t)$ (with respect to the second variable $t\in\mathbb{R}$) satisfies the growth condition

$$|z|\leqslant c_1(1+|t|^{p-1}),\quad \forall z\in\partial J(x,t)\ \text{ a.e. } x\in\Omega,\ \forall t\in\mathbb{R}, \qquad (18)$$

with constants $c_1\geqslant 0$ and $2<p<2^*$. Let α: $H_0^1(\Omega)\to\mathbb{R}\cup\{+\infty\}$ be a convex, lower semicontinuous, proper function. Suppose that

(k$_1$) D_α is closed and there exists $\delta>0$ such that

$$\{v_1+v_2\in H_0^1(\Omega):\ v_1\in V,\ v_2\in V^\perp, \|v_1\|<\delta\}\subset D_\alpha;$$

(k_2) there exists $0<\rho\leqslant\delta$, for $\delta>0$ given in (k_1), such that

$$\int_\Omega J(x,v(x))\mathrm{d}x+\alpha(v)\leqslant 0,\quad \forall v\in V,\quad \|v\|\leqslant\rho;$$

(k_3) $\frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)\|v\|^2+\int_\Omega J(x,v(x))\mathrm{d}x+\alpha(v)\geqslant 0,\ \forall v\in V^\perp$;

(k_4) $\liminf_{\substack{\|v_2\|\to+\infty\\ v_2\in V^\perp}}\frac{1}{\|v_2\|^2}[\int_\Omega J(x,v_1(x)+v_2(x))\mathrm{d}x+\alpha(v_1+v_2)]>-\frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)$ uniformly with respect to $v_1\in V$ on bounded sets in V.

We state the following resonant problem (at the kth eigenvalue λ_k of $-\Delta$ on $H_0^1(\Omega)$).
(P_2) Find $u\in D_\alpha\subset H_0^1(\Omega)$ such that

$$\begin{aligned}&-\int_\Omega\nabla u(x)\cdot\nabla(v-u)(x)\mathrm{d}x+\lambda_k\int_\Omega u(x)(v(x)-u(x))\mathrm{d}x\\&\quad\leqslant\int_\Omega J^0(x,u(x);v(x)-u(x))\mathrm{d}x+\alpha(v)-\alpha(u),\ \forall v\in D_\alpha.\end{aligned}$$

In the statement of (P_2) the notation J^0 stands for the generalized directional derivative of J (in the sense of Clarke [5]) with respect to the second variable.
Our result concerning problem (P_2) is given below.

THEOREM 4. *Assume that conditions* (k_1)–(k_4) *are fulfilled. Then problem* (P_2) *has at least a solution* $u\in H_0^1(\Omega)$ *satisfying* $u\in V^\perp$. *In addition, we have*

$$(1/2)(\|u\|^2-\lambda_k\|u\|^2_{L^2(\Omega)})+\int_\Omega J(x,u(x))\mathrm{d}x+\alpha(u)=0.$$

Proof. We introduce the functional $f=\Phi+\alpha$: $H_0^1(\Omega)\to\mathbb{R}\cup\{+\infty\}$, where Φ: $H_0^1(\Omega)\to\mathbb{R}$ is given by

$$\Phi(v)=\frac{1}{2}(\|v\|^2-\lambda_k\|v\|^2_{L^2(\Omega)})+\int_\Omega J(x,v(x))\mathrm{d}x,\ \forall v\in H_0^1(\Omega). \tag{19}$$

Due to the growth condition (18) we have that Φ in (19) is locally Lipschitz, so f complies with (H_f).
Define

$$Q=\overline{B}_\rho\cap V,\quad S=V^\perp,$$

with $\rho>0$ in (k_2), where $\overline{B}_\rho$ is the closed ball in $H_0^1(\Omega)$ centered at 0 and of radius ρ. Since V is finite dimensional, Q is a compact topological manifold which links with the closed set S as required in (6) (see Rabinowitz [11], p. 24).

Each $u\in Q$ can be expressed as $u=\sum_{i=1}^{k} t_i\varphi_i$, with $t_1,\dots,t_k\in\mathbb{R}$. By (19) and (k_2), we have

$$f(u)=\frac{1}{2}\sum_{i=1}^{k}\left(1-\frac{\lambda_k}{\lambda_i}\right)t_i^2+\int_\Omega J(x,u(x))\mathrm{d}x+\alpha(u)\leqslant 0,\ \forall u\in Q.$$

Thus (f_1) in Theorem 1 holds true.

Every $u\in S$ can be written as $u=\sum_{i=k+1}^{+\infty} t_i\varphi_i$, with $t_i\in\mathbb{R}$, $\forall i\geqslant k+1$. Using (19) and (k_3), it results that

$$\begin{aligned}f(u)&=\frac{1}{2}\sum_{i=k+1}^{+\infty}\left(1-\frac{\lambda_k}{\lambda_i}\right)t_i^2+\int_\Omega J(x,u(x))\mathrm{d}x+\alpha(u)\\&\geqslant\frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)\|u\|^2+\int_\Omega J(x,u(x))\mathrm{d}x+\alpha(u)\geqslant 0,\quad\forall u\in S.\end{aligned}$$

Moreover, in virture of (7), it is seen that

$$0\leqslant a\leqslant c=\inf_{\gamma\in\Gamma}\sup_{z\in\gamma(Q)} f(z)\leqslant\sup_{z\in Q} f(z)\leqslant 0.$$

Consequently (f_2) is satisfied with $a=c=0$.

Let us now check condition (f_3) with $a=0$. Let $\{u_n\}\subset H_0^1(\Omega)$ be a sequence such that $d(u_n,S)\to 0$, $f(u_n)\to 0$ and (12) is satisfied for some $\varepsilon_n\to 0^+$. Consider the decomposition $u_n=u_n^1+u_n^2$ with $u_n^1\in V$ and $u_n^2\in V^\perp$. The property $d(u_n,S)\to 0$ implies that the sequence $\{u_n^1\}$ is bounded in $H_0^1(\Omega)$. Then by (19) we infer that

$$f(u_n)\geqslant -C+\frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)\|u_n^2\|^2+\int_\Omega J(x,u_n(x))\mathrm{d}x+\alpha(u_n),\quad\forall n\geqslant 1,$$

for some constant $C>0$. This inequality in conjunction with (k_4) implies the boundedness of $\{u_n^2\}$ in $H_0^1(\Omega)$. Thus the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Passing eventually to a subsequence of $\{u_n\}$, denoted again $\{u_n\}$, we may admit that $u_n\rightharpoonup u$ in $H_0^1(\Omega)$, $u_n\to u$ in $L^2(\Omega)$ and $u_n(x)\to u(x)$ a.e. $x\in\Omega$. Since D_α is convex and closed (cf. (k_1)), it results that D_α is weakly closed, so $u\in D_\alpha$. Setting $v=u$ in (12) and taking into account relation (2) in [5], p. 77, we deduce

$$\begin{aligned}&\int_\Omega\nabla u_n(x)\cdot\nabla u(x)\mathrm{d}x-\lambda_k\int_\Omega u_n(x)(u(x)-u_n(x))\mathrm{d}x+\\&\quad+\int_\Omega J^0(x,u_n(x);u(x)-u_n(x))\mathrm{d}x+\alpha(u)-\alpha(u_n)\\&\qquad\geqslant-\varepsilon_n\|u_n-u\|+\int_\Omega|\nabla u_n(x)|^2\mathrm{d}x,\quad\forall n\geqslant 1.\end{aligned}$$

By the upper semicontinuity of $J^0(x,\cdot;\cdot)$, Fatou's lemma on the basis of (18) and the lower semicontinuity of α we get $\limsup_{n\to+\infty}\|u_n\|\leqslant\|u\|$. This combined with $u_n\rightharpoonup u$ in $H_0^1(\Omega)$ implies $u_n\to u$ in $H_0^1(\Omega)$. Thereby, (f_3) in Theorem 1 is valid.

Taking $0<\varepsilon_0<\delta$, one obtains from (k_1) that

$$N_{\varepsilon_0}(S)=N_{\varepsilon_0}(V^\perp)\subset\{v_1+v_2\in H_0^1(\Omega)\colon\ v_1\in V, v_2\in V^\perp, \|v_1\|<\delta\}\subset D_\alpha.$$

Finally, for each $l\in]0,\varepsilon_0[$ using the fact $N_\ell(S)\subset N_{\varepsilon_0}(S)\subset \operatorname{int}D_\alpha$ and the continuity of α on $\operatorname{int}D_\alpha$, it results that the set $N_l(S)\cap f^{-\ell}\cap f_l$ is closed. Condition (f_4) is thus satisfied.

Applying Theorem 1 we find a critical point u of f fulfilling $u\in K_0(f)\cap S$. This u solves problem (P_2) (see Clarke [5], pp. 83–85). □

We provide an example where Theorem 4 applies.

EXAMPLE 2. Let a function $J\colon\ \Omega\times\mathbb{R}\to\mathbb{R}$ be measurable with respect to the first variable, locally Lipschitz with respect to the second variable, satisfies the growth condition (18) and

$$-d_1t^2\leqslant J(x,t)\leqslant 0 \quad \text{a.e. } x\in\Omega, \forall t\in\mathbb{R},$$

for some constant $d_1>0$. Let $\alpha\colon\ H_0^1(\Omega)\to\mathbb{R}\cup\{+\infty\}$ be given by

$$\alpha(u)=\begin{cases} d_2\|u_2\|^2 & \text{if} \quad u=u_1+u_2 \text{ with } u_1\in\bar{B}_\delta\cap V \text{ and } u_2\in V^\perp\\ +\infty & \text{otherwise,}\end{cases}$$

with some $\delta>0$ and for a constant $d_2>0$ satisfying

$$\frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)+d_2>\frac{d_1}{\lambda_1}.$$

It is clear that α is convex, lower semicontinuous and proper. We claim that the assumptions of Theorem 4 are verified. Indeed, since $(B_\delta\cap V)\oplus V^\perp\subset D_\alpha$ and D_α is closed, one sees that (k_1) is valid. Condition (k_2) holds with $\rho=\delta$ because α vanishes on $\bar{B}_\delta\cap V$. The estimate

$$\begin{aligned}&\frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)\|v\|^2+\int_\Omega J(x,v(x))\,\mathrm{d}x+\alpha(v)\\ &\quad\geqslant\left[\frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)+d_2-\frac{d_1}{\lambda_1}\right]\|v\|^2\geqslant 0, \quad \forall v\in V^\perp,\end{aligned}$$

ensures that (k_3) is verified according to the choice of d_2. Moreover, we have

$$\int_\Omega J(x, v_1(x)+v_2(x))\mathrm{d}x+\alpha(v_1+v_2)\geqslant -d_1\|v_1+v_2\|^2_{L^2(\Omega)}+d_2\|v_2\|^2$$

$$\geqslant -\frac{d_1}{\lambda_1}\|v_1\|^2+\left(d_2-\frac{d_1}{\lambda_1}\right)\|v_2\|^2,\ \forall v_1\in V,\ v_2\in V^\perp.$$

Thus

$$\liminf_{\substack{\|v_2\|\to+\infty\\ v_2\in V^\perp}}\frac{1}{\|v_2\|^2}\left[\int_\Omega J(x, v_1(x)+v_2(x))\mathrm{d}x+\alpha(v_1+v_2)\right]>-\frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)$$

uniformly with respect to v_1 running in bounded subsets of V, which yields (k_4). Therefore Theorem 4 can be applied.

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